ROBUST RESIDUAL GENERATION USING UNKNOWN INPUT OBSERVERS

Robust Model-Based Fault Diagnosis for Dynamic Systems
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3.1 Introduction

- The generation of robust residuals is the most important task in model-based fault diagnosis techniques.

- Uncertain factors in system modeling are considered to act via an unknown input (or disturbance) on a linear system model.

- Although the unknown input vector is unknown, its distribution matrix is assumed known. Based on the information given by the distribution matrix, the unknown input (disturbance) can be decoupled from the residual.

- The principle of the unknown input observer (UIO) is to make the state estimation error decoupled from the unknown inputs (disturbances). In this way, the residual can also be decoupled from each disturbance, as the residual is defined as a weighted output estimation error.

3.2 Theory and Design of Unknown Input Observers

This section deals with the observer design for a class of systems, in which the system uncertainty can be summarized as an additive unknown disturbance term in the dynamic equation described as following:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Ed(t) \\
y(t) &= Cx(t)
\end{align*}
\]  

(3.1)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( y(t) \in \mathbb{R}^m \) is the output vector, \( u(t) \in \mathbb{R}^r \) is the known input vector and \( d(t) \in \mathbb{R}^r \) is the unknown input (or disturbance) vector. \( A, B, C \) and \( E \) are known matrices with appropriate dimensions.
3.2 Theory and Design of Unknown Input Observers

Remarks:

(a) There is no loss of generality in assuming that the unknown input distribution matrix $E$ should be full column rank. When this is not the case, the following rank decomposition can be applied to the matrix $E$:

$$Ed(t) = E_1E_2d(t)$$

where $E_1$ is a full column rank matrix and $E_2d(t)$ can now be considered as a new unknown input.

(b) The term $Ed(t)$ can be used to describe an additive disturbance as well as a number of other different kinds of modeling uncertainties.

• Examples are: noise, interconnecting terms in large scale systems, non-linear terms in system dynamics, terms arise from time-varying system dynamics, linearization and model reduction errors, parameter variations.

Remarks:

(c) The disturbance term may also appear in the output equation, i.e.,

$$y(t) = Ca(t) + Ey_d(t)$$

This case is not considered here because the disturbance term $Ey_d(t)$ in the output equation can be nulled by simply using a transformation of the output signal $y(t)$, i.e.

$$y_E(t) = Ty(t) = TyCa(t) + TyEy_d(t) = TyCa(t)$$

where $TyEy = 0$, if one replaces $y(t)$ and $C$ with $y_E(t)$ and $TyC$, the problem will be equivalent to one without output disturbances.
3.2 Theory and Design of Unknown Input Observers

Remarks:

(d) For some systems, there is a term relating the control input \( u(t) \) in the system output equation, i.e.

\[ y(t) = Cx(t) + Du(t) \]

As the control input \( u(t) \) is known, a new output can be constructed as:

\[ \bar{y}(t) = y(t) - Du(t) = Cx(t) \]

If the output \( y(t) \) is replaced by \( \bar{y}(t) \), the problem will be equivalent to the one without the term \( Du(t) \). For brevity, the term \( Du(t) \) is omitted in this chapter as this does not affect the generality of the discussion on the observer design.

3.2 Theory and Design of Unknown Input Observers

\[
\begin{align*}
\dot{x}(t) & = Ax(t) + Bu(t) + Ed(t) \\
y(t) & = Cx(t) \\
\end{align*}
\]

(3.1)

Definition 3.1 (Unknown Input Observer (UIO)) An observer is defined as an unknown input observer for the system described by Eq.(3.1), if its state estimation error vector \( e(t) \) approaches zero asymptotically, regardless of the presence of the unknown input (disturbance) in the system.
3.2.1 Theory of UIOs

The structure for a full-order observer is described as:

\[
\begin{align*}
\dot{z}(t) &= Fz(t) + TBu(t) + Ky(t) \\
\hat{z}(t) &= z(t) + Hy(t)
\end{align*}
\] (3.2)

where \( \hat{z} \in \mathbb{R}^n \) is the estimated state vector and \( z \in \mathbb{R}^n \) is the state of this full-order observer, and \( F, T, K, H \) are matrices to be designed for achieving unknown input de-coupling and other design requirements. The observer described by Eq.(3.2) is illustrated in Fig.3.1.

Figure 3.1. The structure of a full-order unknown input observer.
3.2.1 Theory of UIOs

When the observer (3.2) is applied to the system (3.1), the estimation error \( \hat{e}(t) = x(t) - \hat{x}(t) \) is governed by the equation:

\[
\begin{align*}
\dot{\hat{e}}(t) &= (A - HCA - K_1C)e(t) + [F - (A - HCA - K_1C)]z(t) \\
&+ [K_2 - (A - HCA - K_1C)H]y(t) \\
&+ [T - (I - HC)]Bu(t) + (HC - I)Ed(t) \\
\end{align*}
\]

where

\[
K = K_1 + K_2
\]

If one can make the following relations hold true:

\[
\begin{align*}
(HC - I)E &= 0 \\
T &= I - HC \\
F &= A - HCA - K_1C \\
K_2 &= FH
\end{align*}
\]

The state estimation error will then be:

\[
\dot{\hat{e}}(t) = F\hat{e}(t)
\]

If all eigenvalues of \( F \) are stable, \( \hat{e}(t) \) will approach zero asymptotically, i.e. \( \hat{e} \to 0 \). This means that the observer (3.2) is an unknown input observer for the system (3.1) according to Definition 3.1. The design of this UIO is to solve Eqs. (3.4) - (3.8) and making all eigenvalues of the system matrix \( F \) be stable. Before we give the necessary and sufficient conditions for the existence of a UIO, two Lemmas are introduced.

**Lemma 3.1** Eq. (3.5) is solvable iff:

\[
\text{rank}(CE) = \text{rank}(E)
\]

and a special solution is:

\[
H^* = E[(CE)^TCE]^{-1}(CE)^T
\]
3.2.1 Theory of UIOs

\[(HC - I)E = 0 \quad (3.5)\]

**Proof: Necessity:** When Eq. (3.5) has a solution \(H\), one has \(HCE = E\) or
\[(CE)^TH^T = E^T\]
i.e., \(E^T\) belongs to the range space of the matrix \((CE)^T\) and this leads to:
\[\text{rank}(E^T) \leq \text{rank}((CE)^T)\]
i.e.
\[\text{rank}(E) \leq \text{rank}(CE)\]
However,
\[\text{rank}(CE) \leq \min\{\text{rank}(C), \text{rank}(E)\} \leq \text{rank}(E)\]
Hence, \(\text{rank}(CE) = \text{rank}(E)\) and the necessary condition is proved.

---

**Sufficiency:** When \(\text{rank}(CE) = \text{rank}(E)\) holds true, \(CE\) is a full column
rank matrix (because \(E\) is assumed to be full column rank), and a left inverse of \(CE\) exists:
\[(CE)^+ = [(CE)^TCE]^{-1}(CE)^T\]
Clearly, \(H = E(CE)^+\) is a solution to Eq. (3.5).
3.2.1 Theory of UIOs

Lemma 3.2 : Let:

\[ C_1 = \begin{bmatrix} C \\ CA \end{bmatrix} \]

then the detectability for the pair \((C_1, A)\) is equivalent to that for the pair \((C, A)\).

Proof :

If \( s_1 \in C \) is an unobservable mode of the pair \((C_1, A)\), we have:

\[ \text{rank}\left( \begin{bmatrix} s_1I - A \\ C_1 \end{bmatrix} \right) = \text{rank}\left( \begin{bmatrix} s_1I - A \\ C \\ CA \end{bmatrix} \right) < n \]

This means that a vector \( \alpha \in \mathbb{C}^n \) will exist such that:

\[ \begin{bmatrix} s_1I - A \\ C \\ CA \end{bmatrix} \alpha = 0 \]

3.2.1 Theory of UIOs

Proof :

This leads to:

\[ \begin{bmatrix} s_1I - A \\ C \end{bmatrix} \alpha = 0 \quad \alpha \quad \text{rank}\left( \begin{bmatrix} s_1I - A \\ C \end{bmatrix} \right) < n \]

That is to say that \( s_1 \) is also an unobservable mode of the pair \((C, A)\).
3.2.1 Theory of UIOs

**Proof:** If \( s_2 \in \mathbb{C} \) is an unobservable mode of the pair \((C, A)\), we have:

\[
\operatorname{rank}\left[ \begin{bmatrix} s_2 I - A & \mathbb{C} \end{bmatrix} \right] < n
\]

This means that a vector \( \beta \in \mathbb{C}^n \) can always be found, such that:

\[
\begin{bmatrix} s_2 I - A \\ \mathbb{C} \end{bmatrix} \beta = 0
\]

This leads to:

\[
(s_2 I - A)\beta = 0 \quad C\beta = 0
\]
\[
C A\beta = C s_2 \beta = s_2 C \beta = 0
\]

\( s_2 \) is also an unobservable mode of the pair \((C_1, A)\).

As the pairs \((C_1, A)\) and \((C, A)\) have the same unobservable modes, their detectability is formally equivalent.

---

3.2.1 Theory of UIOs

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Ed(t) \\
y(t) &= Cx(t)
\end{align*}
\]  \hspace{1cm} (3.1)

\[
\begin{align*}
\dot{z}(t) &= Fz(t) + TBu(t) + Ky(t) \\
\hat{x}(t) &= z(t) + H_y(t)
\end{align*}
\]  \hspace{1cm} (3.2)

**Theorem 3.1** Necessary and sufficient conditions for (3.2) to be a UIO for the system defined by (3.1) are:

(i) \( \operatorname{rank}(CE) = \operatorname{rank}(E) \)

(ii) \((C, A_1)\) is detectable pair, where

\[
A_1 = A - E[(CE)^TCE]^{-1}(CE)^TCA
\]  \hspace{1cm} (3.12)
3.2.1 Theory of UIOs

\[ K = K_1 + K_2 \]  \hspace{1cm} (3.4)

\[ (HC - I)E = 0 \]  \hspace{1cm} (3.5)

\[ T = I - HC \]  \hspace{1cm} (3.6)

\[ F = A - HCA - K_1 C \]  \hspace{1cm} (3.7)

\[ K_2 = FH \]  \hspace{1cm} (3.8)

Proof: Sufficiency: According to Lemma 3.1, the Eq. (3.5) is solvable when condition (i) holds true. A special solution for \( H \) is \( H^* = E[(CE)^TCE]^{-1}(CE)^T \). In this case, the system dynamics matrix is:

\[ F = A - HCA - K_1 C = A_1 - K_1 C \]

which can be stabilized by selecting the gain matrix \( K_1 \) due to the condition (ii). Finally, the remaining UIO matrices described in (3.2) can be calculated using Eqs. (3.4) – (3.8). Thus, the observer (3.2) is a UIO for the system (3.1).

3.2.1 Theory of UIOs

Necessity: Since (3.2) is a UIO for (3.1), Eq. (3.5) is solvable. This leads to the fact that condition (i) hold true according to Lemma 3.1. The general solution of the matrix \( H \) for Eq. (3.5) can be calculated as:

\[ H = E(CE)^+ + H_0[I_m - CE(CE)^+] \]

where \( H_0 \in R^{n \times m} \) is an arbitrary matrix and \( (CE)^+ \) is the left inverse of \( CE \) which is:

\[ (CE)^+ = [(CE)^TCE]^{-1}(CE)^T \]
3.2.1 Theory of UIOs

Substituting the solution for $H$ into Eq. (3.7), the system dynamics matrix $F$ is:

$$F = A - HCA - K_1 C$$

$$= \left[ I_n - E(CE)^+ C \right] A - \left[ K_1 \quad H_0 \right] \left[ I_n - CE(CE)^+ CA \right]$$

$$= \left[ A_1 - \left[K_1 \quad H_0 \right] \left[ \begin{array}{c} C \\ CA_1 \end{array} \right] \right]$$

where

$$\begin{align*}
\overline{K}_1 &= \left[K_1 \quad H_0 \right] \\
\overline{C}_1 &= \left[ \begin{array}{c} C \\ CA_1 \end{array} \right]
\end{align*}$$

Since the matrix $F$ is stable, the pair $(\overline{C}_1, A_1)$ is detectable, and the pair $(C, A_1)$ also is detectable according to Lemma 3.2.

One should note that the number of independent row of the matrix $C$ must not be less than the number of the independent columns of the matrix $E$ to satisfy condition (i).

- That is to say, the maximum number of disturbances which can be decoupled cannot be larger than the number of the independent measurements.

- It is very interesting to note that observer (3.2) will be a simple full-order Luenberger observer by setting $T = I$ and $H = 0$, when $E = 0$ (i.e. no unknown inputs in the system).

- In this situation, condition (i) in Theorem 3.1 is clearly hold true and condition (ii) is simply changed to that of $(C, A)$ being detectable. This is a well known result in the design of a full-order Luenberger observer.
3.2.1 Theory of UIOs

(ii) \((C, A_1)\) is detectable pair, where

\[
A_1 = A - E(CE)^TCE^{-1}(CE)^TCA
\]

Condition (ii) can be verified in terms of the structural properties of the original system. In fact, this condition is equivalent to the condition that the transmission zeros from the unknown inputs to the measurements must be stable, i.e.

\[
\begin{bmatrix}
    sI_n - A & E \\
    C & 0
\end{bmatrix}
\]

is of full column rank for all \(s\) with \(Re(s) \geq 0\). This can be proved as follows:

---

3.2.1 Theory of UIOs

It can be verified that:

\[
\begin{bmatrix}
    I_n - E(CE)^+C & sE(CE)^+ \\
    0 & I_n \\
    E(CE)^+C & -sE(CE)^+
\end{bmatrix}
\begin{bmatrix}
    sI_n - A & E \\
    C & 0
\end{bmatrix} =
\begin{bmatrix}
    sI_n - A_1 & 0 \\
    C & 0 \\
    -E(CE)^+CA & E
\end{bmatrix}
\]

As the first matrix in the left side of the above equation is a full column rank matrix, we have:

\[
\text{rank}
\begin{bmatrix}
    sI_n - A & E \\
    C & 0
\end{bmatrix} =
\text{rank}
\begin{bmatrix}
    sI_n - A_1 & 0 \\
    C & 0 \\
    -E(CE)^+CA & E
\end{bmatrix}
\]

\[
= \text{rank}
\begin{bmatrix}
    sI_n - A_1 \\
    C \\
    -E(CE)^+CA
\end{bmatrix}
\]

\[+ \text{rank}(E)\]
3.2.1 Theory of UIOs

\[ \text{rank} \begin{bmatrix} sI_n - A_1 \\ C \\ -E(CE) + CA \end{bmatrix} + \text{rank}(E) \]

We have assumed that \( E \) is a full column rank matrix. Hence, condition (ii) is equivalent to the case when the matrix of the left side of the above equation is full column rank for all \( s \) with \( \Re(s) \geq 0 \). This is because the condition for pair \((C, A_1)\) to be detectable is equivalent to the following matrix

\[
\begin{bmatrix}
  sI - A_1 \\
  C
\end{bmatrix}
\]

having full column rank for all \( s \) with \( \Re(s) \geq 0 \).

---

From the above analysis, it can be seen that \( K_1 \) is a free matrix of parameters in the design of a UIO.

After \( K_1 \) is determined, other parameter matrices in the UIO can be computed.

The only restriction on the matrix \( K_1 \) is that it must stabilize the system dynamics matrix \( F \).

The matrix \( K_1 \) which stabilizes the matrix \( F \) is not unique due to the multivariable nature of the problem.

That is to say there is still some design freedom left in the choice of \( K_1 \), after unknown input disturbance conditions have been satisfied. In the following sections, this freedom is exploited further to make the diagnostic residual has directional characteristics or minimum variance properties.
3.2.2 Design procedure for UIOs

One of the most important steps in designing a UIO is to stabilize:

\[ F = A_1 - K_1 C \]

by choosing the matrix \( K_1 \), when the pair \((C, A_1)\) is detectable.

If \((C, A_1)\) is not observable, an observable canonical composition procedure (Chen, 1984) should be applied to \((C, A_1)\), which is:

\[
P A_1 P^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{R}^{n_1 \times n_1}
\]

\[
C P^{-1} = \begin{bmatrix} C^* \\ 0 \end{bmatrix}, \quad C^* \in \mathbb{R}^{n \times n_1}
\]

where \( n_1 \) is the rank of the observability matrix for \((C, A_1)\), and \((C^*, A_{11})\) is observable. The choice of the transformation matrix can be found in Appendix D and Chen (1984). If all eigenvalues of \( A_{22} \) are stable, \((C, A_1)\) is detectable and the matrix \( F \) can be stabilized.

\[
F = A_1 - K_1 C = P^{-1} [P A P^{-1} - P K_1 C P^{-1}] P
\]

\[
= P^{-1} \left\{ \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix} - \begin{bmatrix} K_1^1 \\ K_1^2 \end{bmatrix} [C^* \\ 0] \right\} P
\]

\[
= P^{-1} \begin{bmatrix} A_{11} - K_1^1 C^* \\ A_{12} - K_1^2 C^* \\ 0 \\ A_{22} \end{bmatrix} P
\]

where:

\[
K_1 = PK_1 = \begin{bmatrix} K_1^1 \\ K_1^2 \end{bmatrix}
\]
3.2.2 Design procedure for UIOs

As \((C^*, A_{11})\) is observable, \(K_p^1\) can be determined via the pole placement. The matrix \(K_p^2\) can be any matrix, because it does not affect the eigenvalues of \(F\). The design procedure of a UIO is thus given in Table 3.1.

<table>
<thead>
<tr>
<th>Table 3.1. Unknown input observer (UIO) design procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1° Check the rank condition for (E) and (CE): If (\text{rank}(CE) \neq \text{rank}(E)), a UIO does not exist, go to 10°.</td>
</tr>
</tbody>
</table>
| 2° Compute \(H, T\) and \(A_1\):
  \[
  H = E[(CE)^TCE]^{-1}(CE)^T, \quad T = I - HC, \quad A_1 = TA
  \] |
| 3° Check the observability: If \((C, A_1)\) observable, a UIO exists and \(K_1\) can be computed using pole placement, go to 9°. |

3.2.2 Design procedure for UIOs

Table 3.1. Unknown input observer (UIO) design procedure

| 4° Construct a transformation matrix \(P\) for the observable canonical decomposition: To select independent \(n_1 = \text{rank}(W_0)\) \((W_0\) is the observability matrix of \((C, A_1)\)) row vector \(p_1^T, \cdots, p_{n_1}^T\) from \(W_0\), together other \(n - n_1\) row vector \(p_{n_1+1}^T, \cdots, p_n^T\) to construct an non-singular matrix as:
  \[
  P = [p_1, \cdots, p_{n_0}; p_{n_0+1}, \cdots, p_n]^T
  \] |
| 5° Perform an observable canonical decomposition on \((C, A_1)\):
  \[
  PA_1P^{-1} = \begin{bmatrix}
  A_{11} & 0 \\
  A_{12} & A_{22}
  \end{bmatrix}, \quad CP^{-1} = [C^* \ 0]
  \] |
3.2.2 Design procedure for UIOs

Table 3.1. Unknown input observer (UIO) design procedure

6° Check the detectability of \((C, A_1)\): If any one of the eigenvalues of \(A_{22}\) is unstable, a UIO does not exist and go to 10°.

7° Select \(n_1\) desirable eigenvalues and assign them to \(A_{11} - K_p^1 C^*\) using pole placement.

8° Compute \(K_1 = P^{-1} K_p = P^{-1} [(K_p^1)^T (K_p^2)^T]^T\), where \(K_p^2\) can be any \((n - n_1) \times m\) matrix.

9° Compute \(F\) and \(K\): \(F = A_1 - K_1 C, K = K_1 + K_2 = K_1 + FH\).

10° STOP.

3.2.2 Design procedure for UIOs

\[ \dot{x}(t) = A x(t) + B u(t) + E d(t) + \xi(t) \]
\[ \dot{y}(t) = C x(t) \]
\[ \dot{z}(t) = F \dot{z}(t) + K y(t) \]
\[ \hat{z}(t) = (A - H C A - K C) z(t) + [F - (A - H C A - K C) E] \xi(t) \]
\[ + [K - (A - H C A - K C) F] \hat{z}(t) \]
\[ + [T - (I - H C)] u(t) + (H C - T) E \xi(t) \]

where \(K = K_1 + K_2\).

Theorem 1: Necessary and sufficient conditions for the observer (2) to be a UIO for defined system in (1) are (Chen & Patton, 1999):

\[ \text{rank}(CE) = \text{rank}(E) \]

\[ (C, A) \]

is a detectable pair,

where \(A = A - E \ [(CE)^T CE]^{-1} (CE)^T CA\).
3.2.2 Design procedure for UIOs

6 Remark:

Example: Consider the example used in (Wang et al., 1975; Miller and Mukundan, 1982; Yang and Richard, 1988; Hou and Müller, 1962) with the following parameter matrices:

\[
A = \begin{bmatrix}
-1 & 1 & 0 \\
-1 & 0 & 0 \\
0 & -1 & -1
\end{bmatrix} \quad C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad E = \begin{bmatrix}
-1 \\
0 \\
0
\end{bmatrix}
\]

1°: It can easily be checked that \(\text{rank}(CE) = \text{rank}(E) = 1\).

2°: The matrices \(H\), \(T\) and \(A_1\) are calculated as:

\[
H = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \quad T = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad A_1 = \begin{bmatrix}
0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & -1 & -1
\end{bmatrix}
\]
3.2.2 Design procedure for UIOs

Example:

3°: The pair $(C, A_1)$ is observable, a UIO exists, and the matrix $K_1$ can be determined via the pole placement procedure.

$$K_1 = \begin{bmatrix} 1 & 2 \\ -1 & -6 \\ 0 & 4 \end{bmatrix}$$
which assigns eigenvalues at: $\{-1, -2, -3\}$

Note that the gain matrix $K_1$ is not unique for assigning the same set of eigenvalues.

9°: The matrices $F$ and $K$ are calculated as:

$$F = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 6 \\ 0 & -1 & -5 \end{bmatrix} \quad \quad K = \begin{bmatrix} 0 & 2 \\ -1 & -6 \\ 0 & 4 \end{bmatrix}$$

3.3 Robust Fault Detection and Isolation Schemes based on UIOs

3.3.1 Robust fault detection schemes based on UIOs

The main task of robust fault detection is to generate a residual signal which is robust to the system uncertainty.

A system with possible sensor and actuator faults can be described as:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t) + Bf_a(t) \\ y(t) = Cx(t) + f_s(t) \end{cases}$$

(3.13)

where $f_a \in \mathbb{R}^r$ denotes the presence of actuator faults and $f_s \in \mathbb{R}^m$ denotes sensor faults.
3.3.1 Robust fault detection schemes based on UIOs

When the state estimation is available, the residual can be generated as:

\[ r(t) = y(t) - C\hat{x}(t) = (I - CH)y(t) - C\hat{x}(t) \]  \hspace{1cm} (3.14)

When this UIO-based residual generator applied to the system described in Eq.(3.13), the residual and the state estimation error \( e(t) \) will be:

\[
\begin{align*}
    e(t) &= (A_1 - K_1C)e(t) + TF_s(t) - K_1f_s(t) - H\dot{f}_s(t) \\
    r(t) &= Ce(t) + f_s(t)
\end{align*}
\]  \hspace{1cm} (3.15)

From Eq.(3.15), it can be seen that the disturbance effects have been de-coupled from the residual. To detect actuator faults, one has to make:

\[ TB \neq 0 \]

More specifically, the fault in the \( i_{th} \) actuator will affect the residual \( iff \):

\[ Tb_i \neq 0 \]

3.3.1 Robust fault detection schemes based on UIOs

where \( b_i \) is the \( i_{th} \) column of the matrix \( B \). Similarly, the residual has to be made sensitive to \( f_s(t) \) if sensor faults are to be detected. This condition is normally satisfied, as the sensor fault vector \( f_s(t) \) has a direct effect on the residual \( r(t) \). The robust residual can be used to detect faults according to a simple threshold logic:

\[
\begin{align*}
    \|r(t)\| &< \text{Threshold} \quad \text{for fault-free case} \\
    \|r(t)\| &\geq \text{Threshold} \quad \text{for faulty cases}
\end{align*}
\]  \hspace{1cm} (3.16)
3.3.2 Robust fault isolation schemes based on UIOs

The fault isolation problem is to locate the fault, i.e., to determine in which sensor (or actuator) the fault has occurred.

One of the approaches to facilitate fault isolation is to design a structured residual set.

The term “structured” here means that each residual is designed to be sensitive to a certain group of faults and insensitive to others.

The ideal situation is to make each residual only sensitive to a particular fault and insensitive to all other faults. However, this ideal situation is normally difficult to achieve.

Even when the ideal situation can be achieved, the design freedom will be used up and no freedom will be left for achieving robustness.

3.3.2.1 Robust sensor fault isolation schemes
3.3.2.1 Robust sensor fault isolation schemes.

To design robust sensor fault isolation schemes, all actuators are assumed to be fault-free and the system equations can be expressed as:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Ed(t) \\
y^j(t) &= C^j x(t) + f^j(t) \\
y_j(t) &= c_j x(t) + f_{s_j}(t)
\end{align*}
\]

for \( j = 1, 2, \ldots, m \) (3.17)

where \( c_j \in \mathbb{R}^{1 \times n} \) is the \( j \)th row of the matrix \( C^j \in \mathbb{R}^{(m-1) \times n} \) is obtained from the matrix \( C \) by deleting \( j \)th row \( c_j \). \( y_j \) is the \( j \)th component of \( y \) and \( y_j^i \in \mathbb{R}^{n-1} \) is obtained from the vector \( y \) by deleting \( j \)th component \( y_j \).

3.3.2.1 Robust sensor fault isolation schemes.

Based on this description, \( m \) UIO-based residual generator can be constructed as:

\[
\begin{align*}
\dot{z}^j(t) &= F^j z^j(t) + T^j Bu(t) + K^j y_j^i(t) \\
y^j(t) &= (I - C^j H^j) y_j^i(t) - C^j z^j(t)
\end{align*}
\]

for \( j = 1, 2, \ldots, m \) (3.18)

where the parameter matrices must satisfy the following equations:

\[
\begin{align*}
H^j C^j E &= E \\
T^j &= I - H^j C^j \\
F^j &= T^j A - K_2^j C^j \\
&= F^j H^j \\
K_2^j &= F^j H^j \\
K^j &= K_2^j + K_2^j
\end{align*}
\]

for \( j = 1, 2, \ldots, m \) (3.19)
3.3.2.1 Robust sensor fault isolation schemes

it is clear that each residual generator is driven by all inputs and all but one outputs. When all actuators are fault-free and a fault occurs in the $j_{th}$ sensor, the residual will satisfy the following isolation logic

$$ \begin{align*}
   \| r^j(t) \| & < T^j_{SFI} \\
   \| r^k(t) \| & \geq T^k_{SFI} \quad \text{for} \quad k = 1, \cdots, j-1, j+1, \cdots, m
\end{align*} $$

(3.20)

where $T^j_{SFI}$ ($j = 1, \cdots, m$) are isolation thresholds.

A robust and UIO-based sensor fault isolation scheme is shown in Fig.3.2.
3.3.2.2 Robust actuator fault isolation schemes.

To design robust actuator fault isolation schemes, all sensors are assumed to be fault-free and the system equation can be described as:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B^i u^i(t) + B^i f^i_A(t) + b_i(t_i(t) + f_{ai}(t)) + Ed(t) \\
        &= Ax(t) + B^i u^i(t) + B^i f^i_A(t) + E^i d^i(t) \\
y(t) &= Cx(t) \\
\end{align*}
\]

(3.21)

where \( b_i \in \mathbb{R}^n \) is the \( i \)th column of the matrix \( B \), \( B^i \in \mathbb{R}^{nx(r-1)} \) is obtained from the matrix \( B \) by deleting the \( i \)th column \( b_i \), \( u_i \) is the \( i \)th component of \( u \), \( u^i \in \mathbb{R}^{r-1} \) is obtained from the vector \( u \) by deleting the \( i \)th component \( u_i \), and

\[
E^i = [E \ b_i] \quad \quad d^i(t) = \begin{bmatrix} d(t) \\ u_i(t) + f_{ai}(t) \end{bmatrix} \quad \text{for} \quad i = 1, 2, \ldots, r
\]

3.3.2.2 Robust actuator fault isolation schemes.

Based on the above system description, \( \tau \) UIO-based residual generators can be constructed as:

\[
\begin{align*}
\dot{z}^i(t) &= F^i z^i(t) + T^i B^i u^i(t) + K^i y(t) \\
r^i(t) &= (I - C H^i) p(t) - C z^i(t) \quad \text{for} \quad i = 1, 2, \ldots, r
\end{align*}
\]

(3.22)

The parameter matrices must satisfy the following equations:

\[
\begin{align*}
H^i C^i &= E^i \\
T^i &= I - H^i C \\
F^i &= T^i A - K^i C \quad \text{to be stabilized} \\
K^i_1 &= F^i H^i \\
K^i &= K^i_1 + K^i_2
\end{align*}
\]

(3.23)
3.3.2.2 Robust actuator fault isolation schemes.

One can see that each residual generator is driven by all outputs and all but one inputs. When all sensors are fault-free and a fault occurs in the $i_{th}$ actuator, the residual will satisfy the following isolation logic:

$$
\begin{align*}
\|r^i(t)\| < T_{AFI}^i & \quad \text{for } k = 1, \ldots, i-1, i+1, \ldots, n \\
\|r^k(t)\| & \geq T_{AFI}^k
\end{align*}
$$

(3.24)

where $T_{AFI}^i$ ($i = 1, \ldots, r$) are isolation thresholds.

3.3.2.2 Robust actuator fault isolation schemes.

A robust and VIO-based actuator fault isolation scheme is shown in Fig.3.3.

Figure 3.3. A robust actuator fault isolation scheme.
3.3.2.2 Robust actuator fault isolation schemes.

Remarks:

- The isolation schemes presented in this section can only isolate a single fault in either a sensor or an actuator, at the same time.
- If simultaneous faults need to be isolated, the fault isolation scheme should be modified based on a regrouping of faults.
- Each residual will be designed to be sensitive to one group of faults and insensitive to another group of faults.
- FDI schemes are related to particular systems, a general scheme cannot be expected to suit any system without any modification.

3.3.3 A practical example

Robust actuator fault detection and isolation for (a chemical reactor)

This system is used here to demonstrate the robust actuator fault detection and isolation scheme developed in Section 3.3.2.

3.3.3.1 System representation.

The state, input and output vectors for the considered chemical reactor are:

\[
\begin{align*}
    x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, \\
    u(t) &= \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}, \\
    y(t) &= \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}
\end{align*}
\]

where

\[
\begin{align*}
    C_e(t) &= 3.6C_1(t), \\
    T_e(t) &= 3.6T_1(t), \\
    T_m(t) &= 3.6T_m(t)
\end{align*}
\]
3.3.3 A practical example

3.3.3.1 System representation.

\[ C_o \quad \rightarrow \quad \text{concentration of the chemical product} \]
\[ T_o \quad \rightarrow \quad \text{temperature of the product} \]
\[ T_w \quad \rightarrow \quad \text{temperature of jacket water of heat exchanger} \]
\[ T_m \quad \rightarrow \quad \text{cooler temperature} \]
\[ C_i \quad \rightarrow \quad \text{inlet concentration of reactant} \]
\[ T_i \quad \rightarrow \quad \text{inlet temperature} \]
\[ T_{wi} \quad \rightarrow \quad \text{cooler water inlet temperature} \]

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) + E d(t) \\
y(t) &= C x(t)
\end{align*}
\]

where the term \( Ed(t) \) is used to represent the nonlinearity in the system, and

\[
d(t) = 3.012 \times 10^{12} \exp\left(-\frac{1.2515 \times 10^7}{T_b}\right) = 3.012 \times 10^{12} \exp\left(-\frac{1.2515 \times 10^7}{x_2(t)}\right)
\]

3.3.3 A practical example

3.3.3.1 System representation.

\[
A = \begin{bmatrix}
-3.6 & 0.0 & 0.0 & 0.0 \\
0.0 & -3.6702 & 0.0 & 0.0702 \\
0.0 & 0.0 & -36.2588 & 0.2588 \\
0.0 & 0.6344 & 0.7751 & -1.4125
\end{bmatrix} \quad E = \begin{bmatrix}
1.0 \\
20.758 \\
0.0 \\
0.0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Note that the system matrices are not exactly the same as given by Watanabe and Himmelblau (1982), this is because the time scale has been changed to hours for the sake of convenience.
3.3.3 A practical example

3.3.3.2 UIOs design and residuals generation

UIOs design and residuals generation. Both control inputs $u_1(t)$ ($C_i(t)$) and $u_2(t)$ ($T_i(t)$) are related to the inlet chemical substance, and any fault in $u_1(t)$ or $u_2(t)$ will cause a similar consequence. Hence it is not necessary to isolate faults between $u_1(t)$ and $u_2(t)$. Two UIOs are designed here, the first UIO is driven by $u_1(t)$ and $u_2(t)$ and the second UIO is driven by $u_3(t)$. These two UIOs are robust to the non-linear factor in $d(t)$.

---

UIO 1:
The dynamic equation for the first UIO is:

$$z^1(t) = F^1 z(t) + K^1 y(t) + T^1 [b_1 \ b_2] \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

where $b_1$ and $b_2$ are the first two columns of $B$, and the parameter matrices for this UIO are:

$$H^1 = \begin{bmatrix} 21.758 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} ; \quad T^1 = \begin{bmatrix} -20.758 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 2075.8 & -100.0 & 0.0 & 1.0 \end{bmatrix}$$
3.3.3 A practical example

3.3.3.2 UIOs design and residuals generation

UIO 1:

\[ F^1 = \begin{bmatrix} -10 & 0.0 & 0.0 & 0.0702 \\ 0 & -\lambda_1 & 0.0 & 0.0 \\ 0 & 0.0 & -\lambda_2 & 0.0 \\ 0 & 0.0 & 0.0 & -8.4325 \end{bmatrix} \quad K^1 = \begin{bmatrix} -278.5724 & 13.3496 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 10031.3035 & -475.5956 & 0.7781 \end{bmatrix} \]

The sub-observer for \( z_1 \) and \( z_3 \) (element of vector \( z^1 \)) has no inputs of \( y, u_1 \) and \( u_2 \), and has no coupling with \( z_1 \) and \( z_3 \), hence \( z_1 \) and \( z_3 \) will stay at zero if the initial values of \( z_2 \) and \( z_4 \) are zero and the observer matrix \( F^1 \) is designed to be stable.

\[ z_1^1 = \begin{bmatrix} z_1^1 + 21.758y_1 - y_2 \\ y_2 \\ y_3 \\ z_4^1 - 2075.8y_1 + 100y_2 \end{bmatrix} \]

The residual is generated by:

\[ r^1(t) = y_1(t) - \hat{y}_1(t) = y_1(t) - \hat{x}_1(t) = y_2(t) - z_4^1(t) - 20.758y_1(t) \]
3.3.3 A practical example

3.3.3.2 UIOs design and residuals generation

UIO 2: The dynamic equation for the second VIO is:

\[ \dot{x}^2(t) = F^2 x(t) + K^2 y(t) + T^2 b_3 u_3(t) \]

where \( b_3 \) is the third column of \( B_3 \), and the parameter matrices for this UIO are:

\[
H^2 = \begin{bmatrix}
1.0 & 0.0 & 0.0 \\
0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 40.0 \\
0.0 & 0.0 & 0.0
\end{bmatrix};
\]

\[
T^2 = \begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & -40.0 & 0.0
\end{bmatrix}
\]

\[
F^2 = \begin{bmatrix}
-\lambda_1 & 0.0 & 0.0 & 0.0 \\
0.0 & -\lambda_2 & 0.0 & 0.0 \\
0.0 & 0.0 & -10.0 & 0.2588 \\
0.0 & 0.0 & 0.0 & -11.7645
\end{bmatrix};
\]

\[
K^2 = \begin{bmatrix}
0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -15.9068 \\
0.0 & 0.0 & 0.0
\end{bmatrix}
\]

3.3.3 A practical example

3.3.3.2 UIOs design and residuals generation

UIO 2: Similar to the first UIO, the UIO 2 can also be reduced as:

\[
\begin{bmatrix}
\dot{x}^2_2 \\
\dot{x}^2_3
\end{bmatrix} = \begin{bmatrix}
-10.0 & 0.2588 & 0.0 \\
0.0 & -11.7645 & 0.0
\end{bmatrix} \begin{bmatrix}
\dot{x}^2_3 \\
\dot{x}^2_4
\end{bmatrix} + \begin{bmatrix}
0.0 & 0.0 & -15.9068 \\
0.0 & 0.0 & 980.5501
\end{bmatrix} y + \begin{bmatrix}
1.0 \\
-40.0
\end{bmatrix} u_3(t)
\]

\[
\dot{x}^2 = [y_3, y_3', x^2_3, x^2_4 + 40y_3]^T
\]

The residual is generated by:

\[ r^2(t) = y_3(t) - \hat{y}_3(t) = y_3(t) - \dot{x}_3(t) = y_3(t) - x^2_3(t) \]
3.3.3 A practical example

3.3.3.2 UIOs design and residuals generation

Simulation:

The above UIOs is applied to the non-linear chemical reaction process to detect and isolate faulty actuators. The system input and the initial state vectors are:

\[ u = \begin{bmatrix} 34.632 \\ 1641.5 \\ 29980 \end{bmatrix} \quad \quad x(0) = \begin{bmatrix} 0.3412 \\ 525.7 \\ 472.2 \\ 496.2 \end{bmatrix} \]

The initial values for UIOs are:

\[ z_1(0) = 518.6174; \quad z_2(0) = -51385.5370; \quad z_3(0) = 472.2; \quad z_4(0) = -18391.8 \]

The sampling interval is set as 0.05 hour, and the simulation is carried out for \( t = 10 \) hours. Various types of faults are introduced to the system at \( t = 4 \)
3.3.3 A practical example

3.3.3.2 UIOs design and residuals generation

Simulation:

- The simulation results are shown in Figs.3.4-3.6.
- From which one can see that the residual is almost zero throughout the 10 hours simulation run for fault-free residuals.
- The residuals of the respective UIO increase in magnitude considerably, when actuator faults occur at t = 4 hours.
- The faults can be easily isolated using the information provided by residuals.

Figure 3.4. UIO residuals when a fault occurs in u_1(t) (without parameter variations)
3.3.3 A practical example

3.3.3.2 UIOs design and residuals generation

Simulation:

Robustness analysis:

From the above analysis and simulation, we know that the fault detection and isolation scheme is robust to nonlinearity in $d(t)$. Robustness with respect to parameter variations is analyzed below.

The system with parameter variations is described as:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) + \sum_{i=1}^{4} I_i w_i(x(t), \Delta A)$$

where: $I_i$ is the $i$th column of identity matrix, $w_i$ represents the variations in $i$th row elements of $A$. 
3.3.3 A practical example

3.3.3.2 UIOs design and residuals generation

Robustness analysis:

This equation can be rewritten as:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) + Ew_1 + I_2(w_2 - 20.758w_3) + I_3w_3 + I_4w_4$$

Parameter variations in the form of $Ew_1$ and $I_3w_3$ will not affect the first UIO, because $T^1E = 0$ and $T^1I_3 = 0$.

Similarly, the parameter variations in the form of $Ew_1$ and $I_2(w_2 - 20.758w_3)$ will not affect the second UIO, because $T^2E = 0$ and $T^2I_2 = 0$.

In all cases, the sensitivities to process parameter variations have been decreased. The robustness of UIOs to process parameter variations can be assessed by the simulation in which the matrix $A$ is changed to:

$$A = \begin{bmatrix}
-4.14 & 0.0 & 0.0 & 0.0 \\
0.0 & -4.22073 & 0.0 & 0.08073 \\
0.0 & 0.0 & -36.4401 & 0.2601 \\
0.0 & 0.9516 & 1.1672 & -2.1188
\end{bmatrix}$$
3.3.3 A practical example

3.3.3.2 UIOs design and residuals generation

Robustness analysis:

The residuals for three types of faults are shown in Figs. 3.7-3.9, from which one can conclude that the robust FDI scheme can reliably detect and isolate faulty actuators even in the presence of process parameter mismatch.

![Figure 3.7](image1)  
**Figure 3.7.** UIO residuals when a fault occurs in $u_1(t)$ (with parameter variations)

![Figure 3.8](image2)  
**Figure 3.8.** UIO residuals when a fault occurs in $u_2(t)$ (with parameter variations)

![Figure 3.9](image3)  
**Figure 3.9.** UIO residuals when a fault occurs in $u_3(t)$ (with parameter variations)
3.3.3 A practical example

Remarks:

- Robust actuator fault detection and isolation based on UIOs has been demonstrated in a chemical reactor example.

- The UIO is a time-invariant linear filter but can also be applied to a class of non-linear time-variant systems if the non-linear function is separated from the linear function and can be treated as an unknown input term.

- The robust FDI based on UIOs has also a certain degree of robustness against parameter variations.

3.4 Robust Fault Detection Filters and Robust Directional Residuals

- Fault detection filters (Beard, 1971) are a particular class of the full-order Luenberger observer with a specially designed feedback gain matrix such that the output estimation error (residual vector) has uni-directional characteristics associated with some known fault directions.

- To be specific, the residual vector of a fault detection filter is fixed along with a predetermined direction for an actuator fault or lies in a specific plane for a sensor fault.

- Since the important information required for isolation is contained in the direction of the residual rather than in its time function, the use of a **Beard Fault Detection Filter (BFDF)** does not require the knowledge of fault modes.
3.4 Robust Fault Detection Filters and Robust Directional Residuals

The fault isolation task can be facilitated by comparing the residual direction with pre-defined fault signature directions (or planes), and only one (or the minimum number of) observers required for fault isolation due to directional characteristics of the residual.

The main drawback of the BFDF is that the robustness problem has not been considered.

This section describes a method to design a robust fault detection filter which is based on the combination of UIO and BFDF theories.

The main principle is that the remaining design freedom, after disturbance decoupling conditions have been satisfied, can be used to make the residual vector have directional characteristics.

3.4.1 Basic principles of fault detection filters

In order to describe the BFDF theory, let us consider a system without disturbances in the state space format as:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + b_if_{ai}(t) \\
y(t) &= Cx(t) + I_jf_{sj}(t)
\end{align*}
\]  

(3.25)

The term \( b_if_{ai}(t) \) \((i = 1, 2, \cdots, r)\) denotes that a fault has occurred in the \( i_{th} \) actuator, \( b_i \in \mathbb{R}^n \) is the \( i_{th} \) column of the input matrix \( B \) and is defined as the fault event vector of the \( i_{th} \) actuator fault, and \( f_{ai}(t) \) is an unknown scalar time-varying function which represents the evolution of the fault. The term \( I_jf_{sj}(t) \) \((j = 1, 2, \cdots, m)\) denotes that a fault occurs in the \( j_{th} \) sensor, \( I_j \in \mathbb{R}^m \) is a unit vector corresponding to a fault with the \( j_{th} \) sensor. Note that component faults appear in the system equation in the same way as the actuator fault and hence are not discussed further here.
3.4.1 Basic principles of fault detection filters

A BFDF is just a full-order observer and its structure and the residual can be described as:

\[
\begin{align*}
\dot{x}(t) &= A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{z}(t)) \\
\tau(t) &= y(t) - C\hat{z}(t)
\end{align*}
\]  

(3.26)

where \( \tau \in \mathbb{R}^m \) is the residual vector, \( \hat{z} \in \mathbb{R}^n \) is the state estimation, \( K \in \mathbb{R}^{m \times n} \) is the observer gain matrix which has to be specially designed to make the residual have restricted uni-directional properties in the presence of a particular fault.

---

3.4.1 Basic principles of fault detection filters

If the state estimation error is defined as: \( e(t) = x(t) - \hat{z}(t) \), the residual and \( e(t) \) will be governed by the following error system, when a fault occurs in the \( i_{th} \) actuator:

\[
\begin{align*}
\dot{e}(t) &= (A - KC)e(t) + b_i f_{a_i}(t) \\
\tau(t) &= Ce(t)
\end{align*}
\]  

(3.27)

When a fault occurs in the \( j_{th} \) sensor, the error system will be:

\[
\begin{align*}
\dot{e}(t) &= (A - KC)e(t) - k_j f_{s_j}(t) \\
\tau(t) &= Ce(t) + I_j f_{s_j}(t)
\end{align*}
\]  

(3.28)

where \( k_j \) is the \( j_{th} \) column of the detection filter gain matrix.
3.4.1 Basic principles of fault detection filters

The task of BFDF design is to make \( Ce(t) \) have a \textit{fixed direction} in the output space responding to either \( b_i f_s(t) \) or \( k_j f_d(t) \). Both actuator and sensor fault situations can be considered in the following general error system equation:

\[
\begin{align*}
\dot{e}(t) &= (A - KC)e(t) + l_i \xi_i(t) \\
r(t) &= Ce(t)
\end{align*}
\]  

(3.29)

where \( l_i \in \mathbb{R}^n \) is called the \textit{fault event direction}. The definition of the isolability of a fault with known direction \( l_i \) is given by Beard (1971) as stated below:

\[\text{Definition 3.2 (Isolability of a fault with a given direction)}\]

The fault associated with \( l_i \) in the system described by Eq. (3.29) is isolable if there exists a filter gain matrix \( K \) such that:

\begin{enumerate}
\item \( r(t) \) maintains a \textit{fixed direction} in the output space, and
\item \( (A - KC) \) can be stabilized.
\end{enumerate}

Condition (a) which guarantees that the residual has uni-directional characteristics, is equivalent to ensuring that the rank of the controllability matrix of \((A, l_i)\) pair is one:

\[\text{rank}[l_i (A - KC) l_i \cdots (A - KC)^{n-1} l_i] = 1\]
3.4.1 Basic principles of fault detection filters

Condition (b) ensures the convergence of the filter.

Condition (b) requires arbitrarily assignment of eigenvalues of \((A - KC)\).

This condition has been modified as the stability requirement is sufficient if the residual response time does not need to specified.

This definition was referred to as "fault detectability".

The term "isolability" is more appropriate, because the directional property of the residuals is especially desirable for fault isolation purposes, although it can also be used for fault detection.

Hence, the BFDF is designed to satisfy the fault isolability.

---

3.4.1 Basic principles of fault detection filters

Here the abbreviation BFDF is reserved for a filter (an observer) with residual having uni-directional properties. If a fault associated with the direction \(b_i\) is isolable, the residual of the BFDF will be fixed in the direction parallel to \(Cb_i\), when a fault occurs in the \(i_{th}\) actuator. Similarly, the residual will lie somewhere in the plane defined by \(Cb_j\) and \(I_j\), when a fault occurs in the \(j_{th}\) sensor.
3.4.1 Basic principles of fault detection filters

To isolate faults associated with $p$ isolaible fault event directions $l_i$ ($i = 1, \cdots, p$), the following output separability condition (Beard, 1971) must be satisfied.

**Definition 3.3 (Output Separability of Faults)** The faults associated with $p$ fault event directions $l_i$ ($i = 1, 2, \cdots, p$) are separable in the residual space if the vectors $Cl_1, Cl_2, \cdots, Cl_p$ are linearly independent.

Output separability is necessary for a group of faults to be isolated in the residual space according to their signature directions. The directions $Cl_i$ ($i = 1, 2, \cdots, p$) are then known as the fault signature directions in the residual space.

---

3.4.1 Basic principles of fault detection filters

**Definition 3.4 (Mutual Isolability)** The faults which associated with the fault event vectors $l_i$ ($i = 1, 2, \cdots, p$) are mutually isolaible if there exists a filter gain matrix $K$ satisfying the isolaibility conditions of Definition 3.2 for all $l_i$ ($i = 1, 2, \cdots, p$), i.e.

$$\text{rank}[l_i \ (A-KC)l_i \ \cdots \ (A-KC)^{n-1}l_i] = 1 \text{ for all } i = 1, 2, \cdots, p$$

A group of mutually isolaible faults can be isolated using the residual generated by a single BFDF by comparing the residual direction with the fault signature directions, when there are no simultaneous faults.

If a group of faults is not mutually isolaible, it can be divided into a number of subgroups and each subgroup is mutually isolaible. For such cases, a few BFDFs are required to fulfill the fault isolation task.

In any case, only a minimum number of filters are required for fault isolation. This is the most important and appealing advantage of the BFDF approaches.
3.4.2 Disturbance de-coupled fault detection filters and robust fault isolation

It can be seen that uncertain factors associated with a dynamical system such as disturbances and modeling errors have not be considered in the design of BFDFs.

Now, consider a system with disturbance term $Ed(t)$ and possible sensor and actuator faults described as:

$$\begin{align*}
\dot{x}(t) &= A x(t) + Bu(t) + Ed(t) + b_f f_a(t) \\
y(t) &= C x(t) + I_j f_a(t) 
\end{align*}$$  \hfill (3.30)

If a standard BFDF described by Eq.(3.26) is applied to such a system, the state estimation error and residual will be:

$$\begin{align*}
\dot{e}(t) &= (A - KC)e(t) + Ed(t) + b_f f_a(t) - k_j f_s(t) \\
r(t) &= Ce(t) + I_j f_s(t) 
\end{align*}$$  \hfill (3.31)

• It is clear from Eq. (3.31) that all faults and disturbances affect the residual. It is not easy to discriminate between faults and disturbances if this residual is used to detect and isolate faults.

• Hence, it is necessary to de-couple disturbance effects from the residual for reliable diagnosis.

• It has been shown that the disturbances can be de-coupled from the state estimation error using an unknown input observer (see also Section 3.3.1)
3.4.2 Disturbance de-coupled fault detection filters and robust fault isolation

using the unknown input observer

The residual is thus defined as:

$$ r(t) = y(t) - C\hat{x}(t) = (I - CH)y(t) - C\hat{z}(t) \quad (3.32) $$

the residual and the state estimation error ($e(t)$) will be:

$$ \begin{cases} \dot{e}(t) = (A_1 - K_1C)e(t) + Tb_i f_{si}(t) \\ r(t) = Ce(t) \end{cases} \quad (3.33) $$

when a fault occurs in the $i_{th}$ actuator.

Similarly,

$$ \begin{cases} \dot{e}(t) = (A_1 - K_1C)e(t) - k_{ij} f_{sj}(t) - h_j f_{sj}(t) \\ r(t) = Ce(t) + I_j f_{sj}(t) \end{cases} \quad (3.34) $$

when a fault occurs in the $j_{th}$ sensor. Where $k_{ij}$ is the $j_{th}$ column of the matrix $K_1$ and $h_j$ is the $j_{th}$ column of the matrix $H$. From Eq.(3.33) & (3.34), it can be seen that the disturbance effects have been de-coupled from the residual. This robust (in the disturbance de-coupling sense) residual can be used to detect faults according to a simple threshold logic:

$$ \begin{cases} ||r(t)|| < \text{Threshold} \quad \text{for fault-free case} \\ ||r(t)|| \geq \text{Threshold} \quad \text{for faulty cases} \end{cases} \quad (3.35) $$
3.4.2 Disturbance de-coupled fault detection filters and robust fault isolation

one has to make the residual generated by a UIO have the directional properties in order to achieve robust fault isolation.

From the design of UIOs, it is known that the matrix $K_1$ can be designed arbitrarily after the robust (in the sense of disturbance de-coupling) conditions have been satisfied.

This design freedom can be exploited to make the residual have the uni-directional property.

Comparing the error system Eq. (3.33) with Eq. (3.27), it can be seen that the actuator fault is expressed in the same way for a UIO or a standard BFDF.

$$\begin{align*}
\dot{e}(t) &= (A - KC)e(t) + b_1 f_{at}(t) \\
r(t) &= Ce(t)
\end{align*} \quad (3.27)$$

$$\begin{align*}
\dot{e}(t) &= (A_1 - K_1 C)e(t) + T b_1 f_{at}(t) \\
r(t) &= Ce(t)
\end{align*} \quad (3.33)$$

the sensor fault is also expressed in a similar way for both the BFDF and UIO, except an extra term $h_j \dot{s}_j(t)$ occurs in the error equation of the UIO. Fortunately, this term can be treated in the same way as an actuator fault.
### 3.4.2 Disturbance de-coupled fault detection filters and robust fault isolation

However, it must be pointed out that the residual will lie in a subspace spanned by vectors $I_j$, $Ch_j$, and $Ch_I$ when the residual unidirectional property has been satisfied. For constant sensor faults, the term $h_j f_s(t)$ will disappear from the error system and the residual will lie in the plane spanned by the vectors $I_j$ and $Ch_{1j}$, this is same as the BFDF.

It is necessary to combine the theory of UIOs with the theory of BFDFs to design a robust (disturbance de-coupled) fault detection filter.

### The design procedure can be summarized as follows:

- Compute the matrices $H$ and $T$ using Eqs. (3.11) & (3.6), to satisfy disturbance de-coupling conditions.
- Compute $A_1$ using Eq.(3.12).
- Compute $K_1$ to satisfy a uni-directional property using the theory of BFDFs.
- Compute the observer gain matrix $K$ using Eqs.(3.8) & (3.4).

\[
K = K_1 + K_2 \quad (3.4)
\]
\[
T = I - HC \quad (3.6)
\]
\[
K_2 = FH \quad (3.8)
\]
\[
H^* = E[(CE)^TCE]^{-1}(CE)^T \quad (3.11)
\]
\[
A_1 = A - E[(CE)^TCE]^{-1}(CE)^TC A \quad (3.12)
\]
The design procedure can be summarized as follows:

- Compute the matrices $H$ and $T$ using Eqs. (3.11) & (3.6), to satisfy disturbance de-coupling conditions.
- Compute $A_1$ using Eq. (3.12).
- Compute $K_1$ to satisfy a uni-directional property using the theory of BFDFs.
- Compute the observer gain matrix $K$ using Eqs. (3.8) & (3.4).

\[
K = K_1 + K_2 \quad (3.4)
\]
\[
T = I - HC \quad (3.6)
\]
\[
K_2 = FH \quad (3.8)
\]
\[
H^* = E[(CE)^TCE]^{-1}(CE)^T \quad (3.11)
\]
\[
A_1 = A - E[(CE)^TCE]^{-1}(CE)^TA \quad (3.12)
\]

The key step is then to design the matrix $K_1$.

Once this matrix is available, the computation of other matrices is very straightforward. The BFDF design procedure can be found in the well known literature and is not presented in this chapter.

To show the basic idea, an ideal situation is discussed now in which the number of independent measurements is equal to the number of states, i.e. rank($C$) = $n$.

All eigenvalues of the matrix $A_1 - K_1C$ can be assigned to the same value $-\sigma < 0$, i.e.,

\[
A_1 - K_1C = -\sigma I
\]

This can be achieved by setting $K_1$ as:

\[
K_1 = (A_1 + \sigma I)C^+ \quad (3.36)
\]
3.4.2 Disturbance de-coupled fault detection filters and robust fault isolation

where $C^*$ is the pseudo-inverse of $C$. For this design, the residual will be:

$$ r(t) = C e(t) + I_j f_a(t) $$

$$ = I_j f_a(t) + C e^{-\sigma(t-t_0)} e(t_0) $$

$$ + C \int_{t_0}^{t} e^{-\sigma(t-\tau)} [I_j f_a(\tau) - h_j f_a(\tau) - h_a(t) + h_a(t)] d\tau $$

$$ = Ce^{-\sigma(t-t_0)} e(t_0) + CTh_i \int_{t_0}^{t} e^{-\sigma(t-\tau)} f_a(\tau) d\tau $$

$$ + I_j f_a(t) - Ck_i \int_{t_0}^{t} e^{-\sigma(t-\tau)} f_a(\tau) d\tau - Ck_j \int_{t_0}^{t} e^{-\sigma(t-\tau)} f_a(\tau) d\tau $$

$$ = Ce^{-\sigma(t-t_0)} e(t_0) + CTh_i \beta(t, t_0) $$

$$ + I_j f_a(t) + Ck_j \beta(t, t_0) + Ch_j \gamma(t, t_0) $$

3.4.2 Disturbance de-coupled fault detection filters and robust fault isolation

Clearly, the residual is parallel to $CTh_i$ after the transient has settled down following a fault in the $i_{th}$ actuator. Similarly, the residual will lie in the subspace spanned by vectors $I_j, Ck_i,$ and $Ch_j$, when a fault occurs in the $j_{th}$ sensor.

Due to the residual directional property, the fault can be isolated by comparing the residual direction with the fault signature directions (or subspaces).

**Definition 3.5** The direction of $CTh_i$ is termed a signature direction of the $i_{th}$ actuator fault (Chen, Patton and Zhang, 1996).

The directional relationship between two vectors $CTh_i$ and $r(t)$ can be quantified by the correlation parameter $CORR_i$:

$$ CORR_i(t) = \frac{\langle (CTh_i) r(t) \rangle}{||CTh_i||_2 ||r(t)||_2} \quad (3.37) $$

If $CORR_i > CORR_k$, the fault is more likely in the $j_{th}$ rather than in the $k_{th}$ actuator.
3.4.2 Disturbance de-coupled fault detection filters and robust fault isolation

**Definition 3.6** The signature subspace of the \( j_{th} \) sensor fault is defined as (Chen, Patton and Zhang, 1996):

\[
R_j = \text{Span}\{I_j, Ck_j, Ch_j\} \tag{3.38}
\]

The relationship between the vector \( r(t) \) with the subspace \( R_j \) can be measured by the relationship between the vector \( r(t) \) with its projection \( r_j^*(t) \) in the subspace \( R_j \). This is quantified by:

\[
\text{CORR}_j(t) = \frac{|(r_j^*)^T r(t)|}{||r_j^*||_2 ||r(t)||_2} \tag{3.39}
\]

where the projection \( r_j^*(t) \) of \( r(t) \) in \( R_j \) is:

\[
r_j^*(t) = \Phi_j (\Phi_j^T \Phi_j)^{-1} \Phi_j^T r(t) \tag{3.40}
\]

where

\[
\Phi_j = [I_j Ck_{1j} Ch_j]
\]

3.4.2 Disturbance de-coupled fault detection filters and robust fault isolation

If \( \text{CORR}_j > \text{CORR}_k \), the fault is more likely in the \( j_{th} \) rather than in the \( k_{th} \) sensor. The relationship between a residual vector with the signature subspace can also be judged by the normalized projection distance which is defined as:

\[
NPD_j = \frac{||r(t) - r_j^*(t)||_2}{||r(t)||_2} \tag{3.41}
\]

when \( NPD_j \) is the smallest one amongst all \( NPD_j \) \((j = 1, 2, \ldots, n)\), the fault is most likely in the \( j_{th} \) sensor. The idea of fault isolation by comparing the residual direction with the signature subspace is illustrated in Fig.3.10.
3.4.2 Disturbance de-coupled fault detection filters and robust fault isolation

![Diagram of fault isolation based on directional residuals]

**Figure 3.10.** Fault isolation based on directional residuals

3.4.3 Robust isolation of faulty sensors in a jet engine system

The detection of sensor faults in jet engine systems is very important and has become an active research field.

A system can be described as:

\[
\begin{align*}
\dot{X}_1(t) &= f_1(X_1, X_2, X_3) \\
\dot{X}_2(t) &= f_2(X_1, X_2, X_3) \\
\dot{X}_3(t) &= 10(U - X_3)
\end{align*}
\]

where:

- \( X_1 = n_L \) \( \rightarrow \) Low pressure rotor speed
- \( X_2 = n_H \) \( \rightarrow \) High pressure rotor speed
- \( X_3 = W_f \) \( \rightarrow \) Main burner fuel flow
- \( U = W_{fs} \) \( \rightarrow \) Fuel flow command
3.4.3 Robust isolation of faulty sensors in a jet engine system

The following linear model is derived:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]

where the state is \( x = [x_1 \ x_2 \ x_3]^T \) and the measurement vector is:

\[
y = [x_1 \ x_2 \ x_3 \ p_2 \ p_4 \ t_4]^T
\]

where

\[
\begin{align*}
P_2 & \quad \text{High pressure compressor discharge pressure} \\
P_4 & \quad \text{Turbine discharge pressure} \\
T_4 & \quad \text{Turbine exit temperature}
\end{align*}
\]

the linear model matrices are:

\[
A = \begin{bmatrix}
-1.5581 & 0.6925 & 0.3974 \\
0.2619 & -2.2228 & 0.2238 \\
0 & 0 & -10
\end{bmatrix} \quad B = \begin{bmatrix}
0 \\
0 \\
10
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0.55107 & 0.13320 & 0.30603 \\
0.55217 & 0.13526 & 0.32912 \\
-0.23693 & -0.23623 & 0.61299
\end{bmatrix}
\]

A BFDF described by Eq.(3.26) is designed to isolate sensor faults. If all eigenvalues of the filter are set to \(-3\), the gain matrix can be determined as \( K = (3I + A)C^+ \) because \( rank(C) = 3 \). The fault isolation scheme is applied to the non-linear simulation model.


3.4.3 Robust isolation of faulty sensors in a jet engine system

A reliable diagnostic scheme should perform well for a wide range of operating conditions, and hence the input is set at $u = 20\%$ in the simulation.

The sensor fault is simulated as 2% offset around.

In the simulation, we only consider the fault in sensor Nos. 1, 2 and 3, i.e.
the low pressure rotor speed sensor,
the high pressure rotor speed sensor,
and the main burner fuel flow sensor.

After the transient has settled down, the normalized projection distances for different faulty situations are shown in Table 3.2.

<table>
<thead>
<tr>
<th>Faulty sensor</th>
<th>No.1</th>
<th>No.2</th>
<th>No.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$NPD_1$</td>
<td>0.37050</td>
<td>0.77783</td>
<td>0.66589</td>
</tr>
<tr>
<td>$NPD_2$</td>
<td>0.34117</td>
<td>0.65527</td>
<td>0.42465</td>
</tr>
<tr>
<td>$NPD_3$</td>
<td>0.96529</td>
<td>0.71161</td>
<td>0.31559</td>
</tr>
</tbody>
</table>

From Table 3.2, it can be seen that the fault in sensor No.1 (or No.3) can be correctly isolated as the corresponding normalized projection distance $NPD_1$ (or $NPD_3$) is the smallest.

the fault in the sensor No.2 will be mis-reported as a fault in sensor No.3 as $NPD_3$ is the smallest amongst all normalized projection distances.

Moreover, the smallest $NPD$ is not significantly different from other $NPD$s, and this could make isolation difficult when there is noise in the system.
### 3.4.3 Robust isolation of faulty sensors in a jet engine system

The example in Table 3.2 illustrates the importance of robustness in fault isolation.

The false isolation problem is possibly caused by the linearized errors, as the fault isolation scheme is based on the linear model and this scheme is applied to the original non-linear system.

To model a system more accurately, one can consider the inclusion of the second order terms in the system dynamic equation as follows:

where the matrices $A$ and $B$ are the same as for the linear model. The term $Ed(x)$ represents modeling errors and the vector $d(x)$ consists of the second order terms of $x(t)$ as:

$$d(x) = [x_1^2 \ x_2^2 \ x_3^2 \ x_1 x_2 \ x_1 x_3 \ x_2 x_3]^T$$

### 3.4.3 Robust isolation of faulty sensors in a jet engine system

The distribution matrix $E$ can be obtained using an identification procedure based on the least-squares method. Given a series of values $u^{(1)}, u^{(2)}, \ldots, u^{(N)}$ for input $u$, we can obtain the corresponding steady responses $x^{(1)}, x^{(2)}, \ldots, x^{(N)}$ and $d^{(1)}, d^{(2)}, \ldots, d^{(N)}$, which are related by the following steady state equations:

$$
\begin{align*}
Ax^{(1)} + Bu^{(1)} + Ed^{(1)} &= 0 \\
Ax^{(2)} + Bu^{(2)} + Ed^{(2)} &= 0 \\
\vdots \\
Ax^{(N)} + Bu^{(N)} + Ed^{(N)} &= 0
\end{align*}
$$
3.4.3 Robust isolation of faulty sensors in a jet engine system

If $N$ is greater than the dimension of $d(x)$, the least-squares estimate of the matrix $E$ is given as:

$$ E^* = (\Gamma^+ \Psi)^T $$

where $\Gamma^+$ is the pseudo-inverse of $\Gamma$ and

$$ \Gamma = \begin{bmatrix} (d(1))^T \\ (d(2))^T \\ \vdots \\ (d(N))^T \end{bmatrix} \quad \Psi = \begin{bmatrix} (Ax(1) + Bu(1))^T \\ (Ax(2) + Bu(2))^T \\ \cdots \\ (Ax(N) + Bu(N))^T \end{bmatrix} $$

From the simulation, the following estimate is obtained:

$$ E^* = \begin{bmatrix} 1.3293 & 3.4440 & 0.1375 & -5.1304 & -1.7826 & -1.8719 \\ 5.6812 & -0.5281 & -0.3385 & -1.5193 & 0.5229 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} $$

$E^*$ is not a full column rank matrix ($\text{rank}(E^*) = 2$) and should be decomposed as $E^* = E_1 E_2$. Here $E_1$ is a full column matrix and will be used in the robust fault detection filter design.

$$ E_1 = \begin{bmatrix} 6.2006 & 2.8639 \\ 4.1048 & -4.3262 \\ 0 & 0 \end{bmatrix} $$

All eigenvalues of the robust fault detection filter are set to -3. Using the design procedure presented in this chapter, with $E$ replaced by $E_1$. 

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3.4.3 Robust isolation of faulty sensors in a jet engine system

The parameter matrices of the robust fault detection filter are as follows:

\[ H = \begin{bmatrix} 0.6117 & -0.1170 & 0 & 0.3215 & 0.3220 & -0.1295 \\ -0.1170 & 0.9382 & 0 & 0.0605 & 0.0623 & -0.1916 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ T = \begin{bmatrix} 0 & 0 & -0.1251 \\ 0 & 0 & 0.0783 \\ 0 & 0 & 1.0000 \end{bmatrix} \]

\[ K = \begin{bmatrix} -0.0708 & 0.0443 & 0.5658 & 0.1400 & 0.1531 & 0.3540 \\ 0.0443 & -0.0277 & -0.3540 & -0.0876 & -0.0958 & -0.2215 \\ 0.5658 & -0.3540 & -1.5229 & -1.1193 & -1.2239 & -2.8297 \end{bmatrix} \]

This robust fault detection filter is also applied to the non-linear simulation model to isolate faults in sensor Nos. 1, 2 and 3.

---

3.4.3 Robust isolation of faulty sensors in a jet engine system

To compare the isolation performance with the BFDF, the system and fault simulation have been set as exactly the same. The normalized projection distances for different faulty situations are shown in Table 3.3.

<table>
<thead>
<tr>
<th>Faulty sensor</th>
<th>No.1</th>
<th>No.2</th>
<th>No.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( NP D_1 )</td>
<td>0.00821</td>
<td>0.86727</td>
<td>0.90677</td>
</tr>
<tr>
<td>( NP D_2 )</td>
<td>0.88025</td>
<td>0.00213</td>
<td>0.56602</td>
</tr>
<tr>
<td>( NP D_3 )</td>
<td>0.89433</td>
<td>0.02092</td>
<td>0.00159</td>
</tr>
</tbody>
</table>

Performance with the BFDF, the system and fault simulation have been set as exactly the same.

From Table 3.3, one can see that \( NP D_i \) \((i = 1, 2, 3)\) is the smallest one amongst all normalized projection distances when a fault occurs in the \( i \)th sensor. Moreover, the smallest NP is significantly different from other NPDs. This simulation shows that the fault can be correctly isolated using a robust fault detection filter, even in the presence of modeling errors.
3.4.3 Robust isolation of faulty sensors in a jet engine system

Remarks

• This section has studied the design of a robust fault detection filter.
• its application in the sensor fault isolation problem for a jet engine control system.
• The jet engine is a highly non-linear system, and hence the linearization error causes unreliable isolation if the robustness issues are not considered at the design.
• Based on this model, a robust fault detection filter is designed and applied to the non-linear jet engine simulation model and the results show the effectiveness of the robust fault isolation strategy developed in the paper.
• The technique can be applied to the robust fault isolation for a wide range of systems with uncertain factors.

3.5 Filtering and Robust Fault Diagnosis of Uncertain Stochastic Systems

• The problem of detecting and isolating faults in systems with both modeling uncertainty (including unknown disturbances and modeling errors) and noise has not attracted enough research attention.
• Most systems actually suffer from both modeling uncertainty and noise.
• This section proves that the remaining design freedom, after disturbance decoupling, can be utilized to ensure that the state estimation has the required minimal variance when noise (with known statistics) acts upon the system.
• The optimal observer proposed in this section is applied to the robust fault diagnosis problem.
• To detect and isolate faults, the output estimation error is used as a residual which is robust against unknown disturbances and has minimal variance.
3.5.1 Optimal observers for systems with unknown disturbances and noise

Consider the following discrete-time mathematical description of the system:

\[
\begin{align*}
x_{k+1} &= A_k x_k + B_k u_k + E_k d_k + \zeta_k \\
y_k &= C_k x_k + \eta_k
\end{align*}
\]

where \( x_k \in \mathbb{R}^n \) is the state vector, \( y_k \in \mathbb{R}^m \) is the output vector, \( u_k \in \mathbb{R}^r \) is the known input vector and \( d_k \in \mathbb{R}^s \) is the disturbance (or unknown input) vector, \( \zeta_k \) and \( \eta_k \) are independent zero mean white noise sequences with covariance matrices \( Q_k \) and \( R_k \). \( A_k, B_k, C_k \) and \( E_k \) are known matrices with appropriate dimensions.

The term \( E_k d_k \) can be used to describe a number of different kinds of modeling uncertainties:

- Interconnecting terms in the large scale systems, nonlinear terms in system dynamics.
- Linearization and model reduction errors and parameter variations.

In order to estimate the state of the stochastic system with unknown disturbances described by Eq.(3.42), an optimal observer with the following structure is proposed:

\[
\begin{align*}
x_{k+1} &= F_{k+1} x_k + T_{k+1} B_k u_k + K_{k+1} y_k \\
y_{k+1} &= z_{k+1} + H_{k+1} y_{k+1} 
\end{align*}
\] (3.43)

where the matrices \( F_{k+1}, T_{k+1}, K_{k+1} \) and \( H_{k+1} \) are to be designed to achieve disturbance de-coupling minimum variance estimation.
3.5.1 Optimal observers for systems with unknown disturbances and noise

The block diagram to illustrate this optimal observer is shown in Fig. 3.11:

\[ e_{k+1} = z_{k+1} - \left( z_{k+1} - H_{k+1} y_{k+1} \right) \]
\[ = (I - H_{k+1} C_{k+1}) z_{k+1} - H_{k+1} \eta_{k+1} \]
\[ = (I - H_{k+1} C_{k+1}) z_{k+1} - H_{k+1} \eta_{k+1} - F_{k+1} z_{k} + T_{k+1} B_{k} u_{k} + (K_{k+1}^1 + K_{k+1}^2) y_{k} \]
\[ = (I - H_{k+1} C_{k+1}) z_{k+1} - H_{k+1} \eta_{k+1} - T_{k+1} B_{k} u_{k} - F_{k+1} (z_{k} - e_{k} - H_{k} y_{k}) - K_{k+1}^1 (C_{k} z_{k} + \eta_{k}) - K_{k+1}^2 y_{k} \]
\[ = F_{k+1} e_{k} - K_{k+1}^1 \eta_{k} - H_{k+1} \eta_{k+1} \]
\[ + (I - H_{k+1} C_{k+1}) e_{k} - [F_{k+1} - (I - H_{k+1} C_{k+1}) A_{k} + K_{k+1}^1 C_{k}] x_{k} \]
\[ + (I - H_{k+1} C_{k+1}) E_{k} d_{k} - [K_{k+1}^2 - F_{k+1} H_{k}] y_{k} \]
\[ - [T_{k+1} - (I - H_{k+1} C_{k+1})] B_{k} u_{k} \]

\[ (3.44) \]
3.5.1 Optimal observers for systems with unknown disturbances and noise

where

\[ K_{k+1} = K_{k+1}^1 + K_{k+1}^2 \]

If one can make the following relations hold true:

\[
\begin{align*}
E_k &= H_{k+1} C_{k+1} E_k \\
T_{k+1} &= I - H_{k+1} C_{k+1} \\
F_{k+1} &= A_k - H_{k+1} C_{k+1} A_k - K_{k+1}^1 C_k \\
K_{k+1}^2 &= F_{k+1} H_k
\end{align*}
\]

The estimation error will be:

- That is to say, the state estimation will approach the real state asymptotically, in the mean sense.

From Eq. (3.50), it can be seen that the unknown disturbance vector has been decoupled once Eqs. (3.46)-(3.49) hold true.

Loosely speaking, if the matrix \( F_{k+1} \) is stable, \( \mathbb{E}\{e_k\} \to 0 \) and \( \mathbb{E}\{\hat{e}_k\} \to \mathbb{E}\{x_k\} \) (where \( \mathbb{E}\{\cdot\} \) denotes the expectation or mean operator).

- That is to say, the state estimation will approach the real state asymptotically, in the mean sense.

From Eq. (3.50), it can be seen that the unknown disturbance vector has been decoupled once Eqs. (3.46)-(3.49) hold true.

To design the disturbance decoupled observer, one needs to choose the matrix \( H_{k+1} \) to satisfy Eq. (3.46) and to choose the matrix \( K_{k+1}^1 \) to stabilize the matrix \( F_{k+1} \). Once \( H_{k+1} \) and \( K_{k+1}^1 \) have been chosen, other matrices can be determined using Eqs. (3.47) to (3.49).
3.5.1 Optimal observers for systems with unknown disturbances and noise

\[ E_k = H_{k+1} C_{k+1} E_k \]  \hspace{1cm} (3.46)

**Lemma 3.3**

The necessary and sufficient condition for the existence of a solution to Eq. (3.46) is:

\[ \text{rank}(C_{k+1} E_k) = \text{rank}(E_k) \]  \hspace{1cm} (3.51)

Eq. (3.51) is the only condition for achieving disturbance (unknown input) de-coupling.

To satisfy this equation, the number of independent rows of the matrix \( C_{k+1} \) must not be less than the number of independent columns of the matrix \( E_k \).

That is to say, the maximum number of disturbances which can be de-coupled cannot be larger than the number of independent measurements.

---

3.5.1 Optimal observers for systems with unknown disturbances and noise

\[ E_k = H_{k+1} C_{k+1} E_k \]  \hspace{1cm} (3.46)

The general solution for Eq. (3.46) can be constructed as:

\begin{align*}
H_{k+1} & = H_{k+1}^0 + H_{k+1}^1 H_{k+1}^2 \\
H_{k+1}^0 & = E_k (C_{k+1} E_k)^+ \\
H_{k+1}^2 & = I_m - (C_{k+1} E_k)(C_{k+1} E_k)^+ 
\end{align*}  \hspace{1cm} (3.52-3.54)

and \( H_{k+1}^1 \in \mathbb{R}^{n \times m} \) can be arbitrarily chosen. To simplify the observer design, the matrix \( H_{k+1}^1 \) can be set zero for most cases, i.e.,

\[ H_{k+1} = E_k (C_{k+1} E_k)^+ \]  \hspace{1cm} (3.55)
3.5.1 Optimal observers for systems with unknown disturbances and noise

The stability (or convergence) of the observer is dependent on the matrix $F_{k+1}$, once the matrix $H_{k+1}$ is obtained, the system dynamic matrix can be determined by:

$$F_{k+1} = A_{k+1} - K_{k+1}^T C_k$$  \hfill (3.56)

where:

$$A_{k+1} = A_k - H_{k+1} C_{k+1} A_k$$  \hfill (3.57)

The matrix $K_{k+1}$ should be designed to stabilize the observer.

On considering the simplest case, i.e., when the system is time-invariant, the matrix $F$ can easily be stabilized using pole placement if the matrix pair is observable $\{C, A_1\}$.

---

3.5.1 Optimal observers for systems with unknown disturbances and noise

It is clearly of interest to know how good the estimate $\hat{x}_k$ is.

The variance of this estimation can be measured using the error covariance matrix $P_k$ defined as:

$$P_k = \mathbb{E} \{ (x - \hat{x}_k) [ (x - \hat{x}_k)^T ] \}$$

$$e_{k+1} = F_{k+1} e_k - K_{k+1}^T h_k - H_{k+1} h_{k+1} + T_{k+1} z_k$$  \hfill (3.50)

From the Eq.(3.50), it is easy to see that the update of the covariance matrix is:

$$P_{k+1} = (A_{k+1} - K_{k+1}^T C_k) P_k (A_{k+1} - K_{k+1}^T C_k)^T$$

$$+ K_{k+1}^T R_k (K_{k+1})^T + T_{k+1} Q_k T_{k+1}^T + H_{k+1} R_{k+1} H_{k+1}^T$$  \hfill (3.59)
3.5.1 Optimal observers for systems with unknown disturbances and noise

The best (optimal) state estimation should have minimal variance.

From Eq. (3.59), it can be seen that the covariance matrix of the estimation error is controlled by the matrix $K_{k+1}^1$.

**Theorem 3.2** To make the state estimation error $e_{k+1}$ have the minimum variance, the matrix $K_{k+1}^1$ should be determined by:

$$K_{k+1}^1 = A_{k+1}^1 P_k C_k^T [C_k P_k C_k^T + R_k]^{-1}$$  \hspace{1cm} (3.60)

**Proof:**

For brevity, some subscripts are omitted in the following proof.

$$P_{k+1} = A_1^1 P_k (A_1^1)^T + T Q_k T^T + H R_{k+1} H^T$$

$$-K_1^1 C P_k (A_1^1)^T - A_1^1 P_k C (K_1^1)^T + K_1^1 [C P_k C^T + R_k] (R_1^T)$$

As $R_k$ is a positive definite matrix, $C P_k C^T + R_k$ is also positive definite and there exists an invertible matrix $S$, such that:

$$S S^T = C P_k C^T + R_k$$

Let $D = A_1^1 P_k C^T [S^T]^{-1}$, the covariance matrix is:

$$P_{k+1} = A_1^1 P_k (A_1^1)^T + H R_{k+1} H^T - D D^T$$

$$+ [K_1^1 S - D] [K_1^1 S - D]^T + T Q_k T^T$$
3.5.1 Optimal observers for systems with unknown disturbances and noise

Proof:

\[ K_{k+1}^{i} = A_{k+1}^{i} P_{k} C_{k}^{T} (C_{k} P_{k} C_{k}^{T} + R_{k})^{-1} \]  \hspace{1cm} (3.60)

To minimize var\{e_{k+1}\} = trace\{P_{k+1}\}, one should make \( K^{i} S - D = 0 \), this leads to Eq. (3.60) and we have that:

\[ P_{k+1} = A_{k+1}^{i} P_{k} A_{k+1}^{i} (A_{k+1}^{i})^{T} + T_{k+1} Q_{k} T_{k+1}^{T} + H_{k+1} R_{k+1} H_{k+1}^{T} \]  \hspace{1cm} (3.61)

where

\[ P'_{k+1} = P_{k} - K_{k+1}^{i} C_{k} P_{k} (A_{k+1}^{i})^{T} \]  \hspace{1cm} (3.62)
### 3.5.1 Optimal observers for systems with unknown disturbances and noise

It is important to note that the optimal filtering algorithm proposed in this section is equivalent to a standard Kalman filter for systems without unknown disturbances, by setting the matrices $H_{k+1} = 0$ and $T_{k+1} = I$ when there is no disturbance, i.e. $E = 0$.

### 3.5.2 Robust residual generation and fault detection

In order to diagnose faults, a fault indicating signal, i.e. residual, can be generated using the output estimation as follows:

$$ r_k = y_k - \hat{y}_k = (I - C_k H_k) y_k - C_k z_k $$

The system with possible actuator and sensor faults can be described as:

$$
\begin{align*}
\begin{cases}
\dot{x}_{k+1} = A_k x_k + B_k u_k + E_k d_k + \zeta_k + B_k f^a_k \\
\dot{y}_k = C_k x_k + \eta_k + f^s_k
\end{cases}
\end{align*}
$$

(3.64)

where $f^a_k \in \mathbb{R}^r$ is the actuator fault vector and $f^s_k \in \mathbb{R}^m$ is the sensor fault vector.
### 3.5.2 Robust residual generation and fault detection

For this system, the state estimation error and the residual are governed by the following equations:

\[
\begin{align*}
    e_k &= F_k e_{k-1} + K_k^1 y_{k-1} - H_k \eta_k + T_k \zeta_{k-1} + K_k^2 f_{k-1}^a \\
    r_k &= C_k e_k + \eta_k + f_k^a
\end{align*}
\]  

(3.65)

It can be seen that the unknown disturbance term \( E_d \) does not affect the residual, i.e. the residual is robust against unknown disturbances.

As the state estimation error \( e_k \) has minimum variance, the residual is also optimal with respect to noise (with assumed statistics).

---

### 3.5.2 Robust residual generation and fault detection

For the residual, the two hypotheses to be tested can be identified as \( H_0 \), the normal mode, and the faulty mode \( H_1 \). Under the normal (no fault) condition, the statistics of the residual are:

\[
H_0 : \begin{cases}
    \mathbb{E} \{ r_k \} = 0 \\
    \text{covariance} \{ r_k \} = W_k = C_k F_k C_k^T + R_k
\end{cases}
\]

When a fault occurs in the system (\( H_1 \)), the statistics of the residual will be different from the normal mode.

**If one assumes**

that the noise sequences \( \zeta_k \) and \( \eta_k \) are Gaussian white, the residual will also have the Gaussian distribution. To construct a detection decision function (the test statistic) \( \lambda_k \):

\[
\lambda_k = r_k^T W_k^{-1} r_k
\]

(3.67)
### 3.5.2 Robust residual generation and fault detection

which is $\chi^2$ distributed with $m$ degrees of freedom ($m$ is the dimension of $\eta_k$). The test for fault detection is then:

$$\left\{ \begin{array}{l} \lambda_k \geq T_D \quad \text{fault} \\ \lambda_k < T_D \quad \text{no fault} \end{array} \right.$$  \hspace{1cm} (3.68)

where the threshold $T_D$ is determined from the $\chi^2$ distribution table and:

$$\text{Probability}(\lambda_k \leq T_D \mid H_0) = P_f$$  \hspace{1cm} (3.69)

where $P_f$ is the probability of false alarm which is given by the designer.

To increase the reliability of statistical testing, a residual sequence over a time window can be used.

### 3.5.3 An illustrative example

The linearized discrete-time model of a simplified longitudinal flight control system is as follows:

$$\begin{cases}
    x_{k+1} & = & A_k x_k + B_k u_k + \zeta_k + \xi_k \\
    y_k & = & C_k x_k + \eta_k
\end{cases}$$

where the state variables are pitch angle $\delta_x$, pitch rate $\omega_z$ and normal velocity $\eta_y$, the control input is elevator control signal. The system parameter matrices are:

$$A_k = \begin{bmatrix}
    0.0044 & -0.1203 & -0.4302 \\
    0.0017 & 0.9902 & -0.0747 \\
    0 & 0.8187 & 0
\end{bmatrix}, \quad B_k = \begin{bmatrix}
    0.4252 \\
    -0.0082 \\
    0.1813
\end{bmatrix}$$

$$C_k = I_{3 \times 3}, \quad x = [\eta_y \omega_z \delta_x]^T$$
3.5.3 An illustrative example

The covariance matrices for input and output noise sequences are: $Q_k = \text{diag}(0.1^2, 0.1^2, 0.01^2)$ and $R_k = 0.1^2I_{3\times3}$. The term $E_k d_k$ is used here to represent the parameter perturbation in matrices $A_k$ and $B_k$:

$$
E_k d_k = \Delta A_k x_k + \Delta B_k u_k \\
= E \begin{bmatrix}
\Delta a_{11} & \Delta a_{12} & \Delta a_{13} \\
\Delta a_{21} & \Delta a_{22} & \Delta a_{23}
\end{bmatrix} x_k + \begin{bmatrix}
\Delta b_1 \\
\Delta b_2
\end{bmatrix} u_k
$$

with

$$
E = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
$$

where, $\Delta a_{i j}$ and $\Delta b_{i}$ ($i = 1, 2; j = 1, 2, 3$) are perturbations in aerodynamic and control coefficients.

3.5.3 An illustrative example

The simulation is used to assess the usefulness of the optimal observer for estimating states. In the simulation, the input and initial conditions are set as $u_k = 10$, $x_0 = 0$ and $P_0 = 0.1^2I_{3\times3}$. The aerodynamic coefficients are perturbed by $\pm 50\%$, i.e. $\Delta a_{i j} = -0.5 a_{i j}$ and $\Delta b_{i} = 0.5 b_{i}$. Fig.3.12–Fig.3.14 shows the absolute values of the state estimation errors.
3.5.3 An illustrative example

The estimation errors achieved by the traditional Kalman filter (not disturbance decoupled) are also shown in the Fig.3.12-Fig.3.14.

**Figure 3.12.** The state estimation error absolute values for $\eta_y$ (ODDO: Optimal Disturbance Decoupling Observer; KF: Kalman Filter)

**Figure 3.13.** The state estimation error absolute values for $\omega_z$ (ODDO: Optimal Disturbance Decoupling Observer; KF: Kalman Filter)

**Figure 3.14.** The state estimation error absolute values for $\delta_z$ (ODDO: Optimal Disturbance Decoupling Observer; KF: Kalman Filter)
3.5.3 An illustrative example

It can be seen that the method developed in this section can give better state estimation, even when the system parameters have large perturbations.

- A number of situations when aerodynamic coefficients have time-varying (e.g. sinusoid function) perturbations (the results are not shown in this section) have also been simulated.

For such cases, the estimation error using the Kalman filter is always divergent even if the perturbation magnitude is very small.

However, the disturbance de-coupling method given in this section can give satisfactory estimation.

This is expected, since the perturbation effects on the estimation error have been de-coupled.

---

Fig. 3.15 shows the detection function $\lambda_k$ when an incipient (small and slow) fault occurs in the sensor for $\delta_z$.

![Figure 3.15](image)

**Figure 3.15.** The fault detection function when a fault occurs in the sensor for $\delta_z$. 

---
3.5.3 An illustrative example

Fig. 3.16 shows the fault detection function $\lambda_k$ when a step fault occurs in the actuator. It can be seen that the faults are detected very reliably by setting a threshold ($T_D$) on the fault detection function.

![Graph showing fault detection function](image)

Figure 3.16. The fault detection function when a fault occurs in the actuator

Remarks:

This section has proposed a systematic approach to designing optimal disturbance de-coupled observers for systems with both unknown disturbance and noise.

This optimal observer is used to estimate the system state and to generate residuals for detecting faults in stochastic uncertain systems.

The method has been applied to detecting sensor and actuator faults in a simplified flight control system and the simulation results show the effectiveness of the method.
3.6 Summary

- The purpose of this chapter has been the introduction of UIO-based robust residual generation methods.
- A full-order UIO structure has been described in this chapter.
- The existence conditions and design procedures for such UIOs have also been introduced and soundly proved.
- When compared with other techniques in designing UIOs, the existence conditions presented in this chapter are very easy to verify.
- This chapter has exploited the remaining freedom to achieve other performance requirements for FDI.
- And has introduced a method to design a robust fault detection filter which can generate disturbance de-coupled directional residuals for fault isolation. This is achieved via a combination of the UIO and the BFDF principles.

Robust FDI based on UIOs have been studied for many years. However, the number of reported applications is very limited.

The main argument is that the unknown input distribution matrix, required for designing UIOs, is actually unknown for most practical systems.